# Dissipationless BCS Dynamics with Large Branch Imbalance

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In many situations a BCS-type superconductor will develop an imbalance between the populations of the hole-like and electron-like spectral branches. This imbalance suppresses the gap. It has been noted by Gal'perin, Kozub and Spivak [6] that at large imbalance, when the gap is substantially suppressed, an instability develops. The analytic treatment of the system beyond the instability point is complicated by the fact that the Boltzmann approach breaks down. We study the short time behavior following the instability, in the collisionless regime, using methods developed by Yuzbashyan et al. [18, 19].

## I. INTRODUCTION

The excitation spectrum of a BCS superconductor consists of an electron-like and a hole-like branch. While the two are equally populated in equilibrium, non-equilibrium states may have a 'branch imbalance'. For example, if a superconductor is placed in an NSN junction as in the experiment of Clarke [1, 2, 3], the injected quasi-particles can be primarily electron-like, and the electron-like branch more heavily populated than the hole-like branch in the steady state. Superconducting wires, where branch imbalance arises at phase-slip centres [4], give another example.

This imbalance suppresses the spectral gap [5]. If large enough, it returns the system to the normal state, which is known to be unstable below $T_c$ – this is the Cooper instability. It was realized by Gal'perin, Kozub and Spivak [6, 7] that the Cooper instability is a limiting case of a more general instability which afflicts any state beyond a critical level of imbalance and results in oscillations of the order parameter. Traditional approaches to the dynamics fail when the instability occurs, and for this reason the supercritical behavior of the superconductor is as yet unknown. It is important to resolve this, since situations in which large imbalance occurs are quite natural, for example in NSN junctions at large enough injection rate, or long superconducting wires held at sufficiently high voltage.

This paper makes a step in this direction by treating the short-time dynamics in the limit where dissipative processes act slowly in comparison with the BCS dynamics (in particular the oscillation of the gap). We work in the regime $T_c - T \ll T_c$, when the slow relaxation of imbalance allows the system to reach a 'quasiequilibrium' whose deviation from equilibrium can be characterized only by the amount of imbalance. We also assume the initial conditions are close to the unstable stationary solution, in a sense defined below.

Since the BCS dynamics are characterized by a time scale of order $1/(T_c - T)$, the assumed separation of scales is $1/(T_c - T) \ll \tau_\epsilon$, where $\tau_\epsilon$ is the time scale associated with energy relaxation. In a metallic superconductor with Debye energy $\Theta \gg T_c$ and $\tau_\epsilon \sim \Theta^2/T_c^3$, this gives only the unrestrictive $(T_c - T)/T_c \gg T_c^2/\Theta^2$.

Often one can avoid the complexities of the microscopic BCS dynamics of a superconductor with a simpler effective description such as the Ginzburg-Landau or the Boltzmann kinetic equation. Two time scales are important in deciding whether either is appropriate: the inelastic quasiparticle relaxation time, $\tau_\epsilon$, and the time scale $\tau_\Delta$ over which the order parameter varies significantly.

When $\tau_\epsilon \ll \tau_\Delta$, the quasiparticle distribution rapidly reaches a local equilibrium characterized by the order parameter $\Delta(\vec{r},t) = |\Delta(\vec{r},t)|e^{i\chi(\vec{r},t)}$, with dynamics described by the Ginzburg-Landau equations for $\Delta(\vec{r},t)$. We are interested in the opposite limit, $\tau_\epsilon \gg \tau_\Delta$, which is usually tackled with the Boltzmann kinetic equation [8] for the quasiparticle distribution function $n(\vec{r},\vec{p},t)$:

$$\frac{\partial n}{\partial t} + \frac{\partial \epsilon}{\partial \vec{p}}\frac{\partial n}{\partial \vec{r}} - \frac{\partial \epsilon}{\partial \vec{r}}\frac{\partial n}{\partial \vec{p}} = -I\{n\}. \tag{1}$$

Here $\epsilon$ is the energy of a quasiparticle state and the functional $I\{n\}$ accounts for impurity scattering and collisions between electrons or between electrons and phonons. The kinetic equation must be supplemented with the self-consistency equation, which determines $|\Delta(\vec{r},t)|$ and thus the quasiparticle energies:

$$1 = \frac{\lambda}{2}\int \frac{1-2n_{\vec{p}}}{\epsilon(\xi_{\vec{p}})}d\xi_{\vec{p}}. \tag{2}$$

Here $\xi_{\vec{p}} = p^2/2m - \mu$, $\mu$ is the chemical potential, and $\lambda$ the BCS coupling constant. In addition there is a 'neutrality condition' involving the phase $\chi$ of the order parameter. This condition arises from the continuity equation and ensures

the conservation of charge – it is independent of (1) in the case of a superconductor. In the spatially homogeneous case, if we define $\Phi \equiv \frac{\hbar}{2}\frac{\partial \chi}{\partial t} + e\varphi$ (where $\varphi$ is the electric potential) and $\tilde{\xi}_{\vec{p}} = \xi_{\vec{p}} + \Phi$, it takes the form

$$\int \frac{n_{\vec{p}}\,\tilde{\xi}_{\vec{p}}}{\epsilon(\xi_{\vec{p}})} d\xi_{\vec{p}} = \Phi. \tag{3}$$

This integral quantifies the branch imbalance. The electron- and hole-like branches are distinguished by the sign of $\tilde{\xi}$. In equilibrium the two branches are identically populated since the quasiparticle energy ($\epsilon^2 = \tilde{\xi}^2 + |\Delta|^2$) is even in this quantity, and $\Phi$ vanishes.

It is important that while branch imbalance is absent in equilibrium, its relaxation rate $\tau_Q$ diverges as $T_c$ is approached [3, 9]: $\tau_Q \sim \tau_\epsilon T_c/\Delta$. (This is in the absence of oscillations of the gap, and assuming relaxation due to electron-phonon collisions [3, 8].) Thus when $\Phi \neq 0$, the quasiparticle distribution reaches a 'quasiequilibrium' on a time of order $\tau_\epsilon$, characterized by distinct chemical potentials for the separately equilibrated hole-like and electron-like branches [3, 5, 9, 10]. This distribution is given below (10). Inserting it into the self-consistency equation (7) reveals that imbalance suppresses the gap relative to $\Delta_0$, its value when $\Phi = 0$:

$$|\Delta|^2 = \Delta_0^2 - 2\Phi^2. \tag{4}$$

At $\Phi = \Delta_0/\sqrt{2}$, the gap is completely suppressed, and the system is returned to the unstable normal state. The instability appears earlier, at an intermediate value $\Phi_c$.

The Boltzmann description cannot handle the system after the instability takes hold, as modes are excited in which Cooper pairs posses non-zero relative phases (the relative phases of the $s_-(\xi) \equiv s_x(\xi) - is_y(\xi)$, in the language of Anderson pseudospins [11]). Only the overall phase of the condensate is retained in the Boltzmann approach, effectively restricting the system to a subclass of solutions where the Cooper pairs precess in phase.

In this situation we must return to the Gorkov equations describing the mean-field dynamics of the individual Cooper pairs [12]. We study here only the limit in which dissipative processes are neglected ($\tau_\epsilon \to \infty$) and the dynamics controlled purely by the BCS Hamiltonian. This was done for the Cooper instability by Barankov, Levitov and Spivak in [13], yielding a 'soliton train' of peaks in the gap. The problem was further discussed by Warner and Leggett in [14], by Barankov and Levitov in [15, 16], and by Yuzbashyan and Dzero in [17]. The integrability of the mean-field BCS system was established by Yuzbashyan, Altshuler, Kuznetsov and Enolskii in the papers [18, 19] and a general framework for addressing its dynamics was developed, which we use here to tackle the more involved case of imbalance. We confine ourselves to the integrable BCS Hamiltonian because it can be treated analytically; it was however shown by Barankov and Levitov in [20] for the case of the Cooper instability that the gap oscillations survive the breaking of integrability.

Our results show the emergence of oscillatory behavior at the instability point. On much larger time scales ($\sim \tau_\epsilon$) dissipation will modulate the form of these oscillations and determine the final fate of the system. Such an analysis is beyond the scope of this paper, but the solutions we present may be thought of as candidates to be found either at very large times after the onset of the instability or at intermediate times on the way to the asymptotic behavior, and are a first step in calculating the long-time behavior. We discuss this briefly in Section V.

The ability to 'switch on' the pairing interaction in ultracold trapped gases via tunable Feshbach resonances [21, 22, 23, 24, 25, 26] means that one might hope to observe the oscillations of the order parameter directly in the case of the Cooper instability [13, 14, 15, 16, 17]. On the other hand, it is not clear whether one could create imbalance in a sufficiently controlled fashion to observe the instability it creates[33]. In contrast, imbalance can be created easily in metals, for example in tunnel junctions [1, 2, 3]. Here the direct observation of the collisionless dynamics is unlikely due to short dissipation times, and its significance is instead in its effect on processes at longer timescales such as the relaxation of imbalance.

The structure of the paper is as follows. Section II presents the equations of motion for the system and the relevant initial conditions, and includes the final result of the analysis. Section III describes the formal solution to BCS dynamics [18, 19, 28, 29], which is applied to the relevant case in Section IV. We conclude in Section V.

## II. GORKOV'S NONLINEAR EQUATIONS & SOLUTION WITH IMBALANCE

The system of equations describing the BCS superconductor in the non-dissipative regime were derived by Volkov and Kogan [12] in the Keldysh Green's function formalism:

$$\dot{\vec{s}}(\xi) = \vec{s}(\xi) \times (2\Delta_x, 2\Delta_y, -2\xi), \tag{5}$$

where $\vec{s}$ is defined by the following Keldysh Green's functions:

$$s_z(\xi) = \langle [c_\uparrow(\xi), c_\uparrow^\dagger(\xi)] \rangle$$
$$s_-(\xi) = \langle [c_\uparrow(\xi), c_\downarrow(\xi)] \rangle \tag{6}$$

$s_- = s_x - i s_y$, and:

$$\Delta = \Delta_x - i\Delta_y = \frac{\lambda}{2} \int s_-(\xi) d\xi. \tag{7}$$

This system of equations can be derived from a classical Hamiltonian:

$$H = \int 2\xi s_z(\xi) d\xi - \frac{2}{\lambda}|\Delta|^2; \qquad \{s_i(\xi), s_j(\xi')\} = \epsilon_{ijk} s_k(\xi) \delta(\xi - \xi'). \tag{8}$$

This is the mean-field BCS Hamiltonian written in terms of Anderson pseudo-spins, with an up (resp. down) spin representing a full (empty) Cooper pair. Singly-occupied pairs decouple from the order parameter dynamics (as can be seen from the BCS Hamiltonian which involves only pair-to-pair scattering) and correspond to zero-length spins. One may derive the Gorkov equations heuristically as a mean-field approximation to the BCS Hamiltonian.

The current paper aims to present solutions of (5) for imbalanced superconductors, using the approach of [18]. Before doing so, let us recall the instability of the stationary solutions of (5) in the presence of imbalance which was pointed out in [6]. Generally, stationary solutions of (5) have the form:

$$\begin{aligned} s_z(\xi) &= -\frac{\xi + \Phi}{\sqrt{(\xi+\Phi)^2 + |\Delta|^2}} (1 - 2n(\xi)) \\ s_-(\xi) &= \frac{\Delta}{\sqrt{(\xi+\Phi)^2 + |\Delta|^2}} (1 - 2n(\xi)) \end{aligned} \tag{9}$$

where $n(\xi)$ is the quasi-particle distribution function, which in the presence of imbalance is[34] [3, 6]

$$n(\xi) = \frac{1}{\exp\left( \frac{\sqrt{(\xi+\Phi)^2+|\Delta|^2} - \Phi \operatorname{sign}(\xi+\Phi)}{T} \right) + 1} \tag{10}$$

which is a Fermi-Dirac distribution for the quasi-particles, with different chemical potentials for the hole-like and electron-like spectral branches implemented by the term $\Phi \operatorname{sign}(\xi + \Phi)$ in the exponent. $\Phi$ parametrizes the amount of imbalance in the system. The expression for $n(\xi)$ is valid when $|\xi + \Phi| \gg |\Delta|$. The self consistency condition (7) is satisfied if [5]:

$$|\Delta|^2 = \Delta_0^2 - 2\Phi^2 \tag{11}$$

where $\Delta_0$ is the order parameter at the temperature $T$ in the absence of imbalance (for the case $\Phi = 0$). Thus imbalance between electron-like and hole-like excitations suppresses the order parameter[35].

One can easily see that the distribution is unstable when $\Phi = \Delta_0/\sqrt{2}$, and $\Delta = 0$ by (11). At this point it becomes

$$n(\xi) = \frac{1}{\exp\left( \frac{\xi \operatorname{sign}(\xi+\Phi)}{T} \right) + 1} \tag{12}$$

which is nothing but the quasi-particle distribution of a normal Fermi gas in the excitation representation, where an artifical distinction between hole-like and electron-like excitations is made at $\xi = -\Phi$. So, the peculiar form of (12) is an artifact of the excitation representation, and it just describes a normal metal placed at $T < T_c$. The Cooper instability of this metal presents itself as a instability of the stationary solution of equation (5).

At $\Phi = 0$ however the solution is stable, and represents the equilibrium superconducting state. There must therefore be an onset of instability at some finite $\Phi$ intermediate between 0 and $\Delta_0/\sqrt{2}$. In order to find this point and in order to give a quantitative characterization of the instability, a linear stability analysis was performed around this solution in [6], looking for the presence of an unstable mode $e^{-i\omega t}$, $\mathrm{Im}(\omega) > 0$. This leads to an integral equation for $\omega$:

$$G(\sqrt{\omega^2/4 - |\Delta|^2}) \equiv \int \frac{d\xi}{\sqrt{\tilde{\xi}^2 + |\Delta|^2}} \frac{1 - 2n(\xi)}{\tilde{\xi} - \sqrt{\omega^2/4 - |\Delta|^2}} = 0. \tag{13}$$

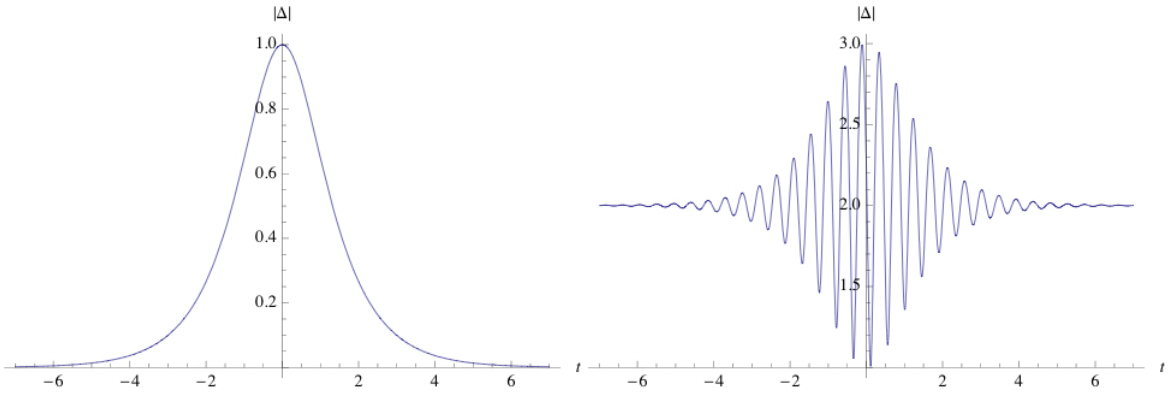


FIG. 1: The soliton for the Cooper instability of a normal metal, with $\gamma = 0.5$, and a pulse of oscillations of $|\Delta(t)|$ for $\Phi = 7$, $\gamma = 0.5$, $\Delta = 2$, showing the qualitative nature of the solution in the two regimes $\Delta = 0$ and $2\gamma < \Delta$. The solution interpolates smoothly between these cases.

Here $\tilde{\xi} = \xi + \Phi$. The solutions of this equation were found to be:

$$\mathrm{Re}(\omega) = 2\Phi, \qquad |\mathrm{Im}(\omega)| = \frac{2}{\pi}((T - T_c) - |\Delta|). \tag{14}$$

The instability arises when the right hand side of this expression for $|\mathrm{Im}(\omega)|$ becomes positive at $\Phi_c$. When the imbalance completely suppresses the gap (14) agrees well with the usual Cooper instability case. In fact all we shall need in the following are the orders of magnitude of the the following quantities:

$$\Delta \sim s^2 T_c, \qquad \gamma \equiv \frac{1}{2}\mathrm{Im}(\omega) \sim s^2 T_c, \qquad \Phi \sim s T_c \tag{15}$$

where $s \equiv \sqrt{(T_c - T)/T_c}$, and we have also defined the parameter $\gamma$, the rate of instability, an important parameter which will appear frequently below. The order of magnitude of the different quantities could just as well have been taken from the Cooper instability case, as they remain the same when the imbalance completely suppresses the gap. Namely these orders of magnitude are not sensitive to the exact form of the distribution function, but rather to gap suppression and may be viewed as consequences of Eq. (11).

The main result of the following analysis is the time dependence of the order parameter following a perturbation from the unstable stationary solution. This is a train of 'solitons' each of the form:

$$\Delta(t) = \Delta\left(1 + \frac{2\gamma}{\Delta} e^{-2i\Phi(t-\chi)} \mathrm{sech}(2\gamma t)\right) \tag{16}$$

where $\Delta$ (as opposed to $\Delta(t)$) denotes the value of the order parameter, which we can take to be real and positive, in the stationary but unstable initial state. The solitons are separated in time by a period that increases logarithmically with the smallness of the perturbation,

$$t_0 = \gamma^{-1} \log\left(\frac{4\gamma}{\delta}\right) \tag{17}$$

in the notation used below. $\chi$ is a real constant (see below) which does not affect the qualitative nature of the soliton.

The behaviour of $|\Delta(t)|$ consists in fast (at frequency $2\Phi$) ocillations within an envelope given by $|1 \pm (2\gamma/\Delta)\mathrm{sech}(2\gamma t)|$. When $2\gamma/\Delta \gg 1$ (recall that as $\Delta \to 0$ we return to the finite-temperature normal metal) this formula gives a soliton of the same shape as that found in [13] for the order parameter oscillations associated with the Cooper instability. In the opposite limit where the instability is small and $2\gamma/\Delta \ll 1$, we find instead pulses of oscillations of $|\Delta(t)|$ during which the value of $|\Delta(t)|$ averaged over over a period of the fast oscillation rises: $|\Delta(t)|_{av} \simeq \Delta(1 + \gamma^2/\Delta^2 \mathrm{sech}^2(2\gamma t))$.

## III. ELEMENTS OF THE INTEGRABLE STRUCTURE

The instability discovered by Gal'perin et al. shows that at large times dissipation may take the system to a new state which is very different to the initial stationary solution. A similar situation exists when a normal metal is placed

at $T < T_c$. For comparison we briefly review here the known results on this instability, the current paper expounding on these results to include the case where imbalance is present.

A normal metal at $T < T_c$ is unstable against the development of a gap (this is the celebrated Cooper instability). The short term dynamics of the system, just after the instability takes hold, consist in an oscillatory behavior of $\Delta$. These oscillations damp out at large times because of dissipation (this process is not described by the pure BCS Hamiltonian) and the material is left in the superconducting state. The oscillatory behavior of $\Delta$ consists of a 'soliton train' which can be found by solving (5) for a system placed near the normal metal state – namely with initial conditions (9), where $n_p$ is the Fermi distribution function plus a small perturbation. This is discussed in [13, 14, 15, 16].

The soliton train describes the short-time ($\ll \tau_\epsilon$) behavior of the order parameter following the onset of the Cooper instability. A similar analysis will be presented here for the case when the initial state is given by (9) and (10), i.e. a superconductor with branch imbalance near $T_c$. We study the behavior of the system after a small perturbation is added to $n(\xi)$. This describes the short time behavior after the instability has taken hold. The approach also yields solutions where the perturbation is rather larger. We believe that these solutions may provide further intuition about the possible routes the system may take once dissipation effects are taken into account.

In order to find these solutions we apply the formalism developed in [18, 19] for the dynamics of the mean field BCS system, which draws heavily on the theory of integrable systems. We shall establish the notation and present the main concepts of the derivation, referring the reader interested in further details to the original papers. In these papers the spectrum is treated as discrete – here we assume a continuous spectrum.

An important object in the integrable structure is the Lax matrix of the system, a $2 \times 2$ matrix depending on a complex parameter $u$, given by:

$$\mathcal{L}(u) = \begin{pmatrix} A(u) & B(u) \\ B^*(u) & -A(u) \end{pmatrix} \tag{18}$$

where:

$$\begin{aligned} A(u) &= \frac{2}{\lambda} - \int \frac{s_z(\xi)}{u-\xi} d\xi, \\ B(u) &= \int \frac{s_-(\xi)}{u-\xi} d\xi. \end{aligned} \tag{19}$$

For any $u$, the matrix $\mathcal{L}(u)$ is time dependent via $\vec{s}$, but its eigenvalues are not – a key to integrability. These eigenvalues constitute an infinite set of constants of motion[36]. They are labeled $v(u)$, and given by:

$$v(u) = \pm\sqrt{-\det \mathcal{L}(u)}. \tag{20}$$

The analytic structure of $v(u)$ is as follows: in general $v(u)$ has branch cuts, with square root behavior around the branch points, parallel to the imaginary $u$ axis, as well as a jump discontinuity on the real axis, on the support of the spectrum. The branch points $E_i$ satisfy $v(E_i) = 0$. Other important quantities are the zeros of $B(u)$, dubbed $u_i$: $B(u_i) = 0$.

An important simplification takes place if one is interested only in the long time behavior of the system (but still at times $\ll \tau_\epsilon$). After an initial transient the system exhibits periodic or quasi-periodic behavior whose frequency is dictated by the branch points of $v(u)$. The jump discontinuity is only relevant for the initial transient. The oscillatory behavior following the transient is captured by a simpler system containing only a finite number of spins $\vec{s}^{(i)}$, $i = 1, ..., g+1$, where $g+1$ is the number of branch cuts in $v(u)$. The system with a finite number of spins is integrable, with a similar integrable structure to the infinite system, integrals being replaced by sums; namely if

$$\begin{aligned} A(u) &= \frac{2}{\lambda} - \sum_{i=1}^{g+1} \frac{s_z^{(i)}}{u-\xi_i} \\ B(u) &= \sum_{i=1}^{g+1} \frac{s_-^{(i)}}{u-\xi_i} \end{aligned} \tag{21}$$

are substituted into (18) then the eigenvalues of the matrix are constants of motion. It is convenient to define $P(u) \equiv \prod_i (u-\xi_i)$, and also a degree $2g+2$ polynomial $Q(u)$ with zeros at $E_i$, through $v(u) = \frac{2}{g}\sqrt{Q(u)}/P(u)$. The appearance of $\sqrt{Q(u)}$ signals the relevance of the algebraic Riemann surface defined by the curve $y(u) = \sqrt{Q(u)}$ to this problem. To find the configuration of the finite number of spins at any given time one must know $Q(u)$, which is independent of time, and the time-dependent quantities $u_i(t)$, defined by $B(u_i(t)) = 0$.

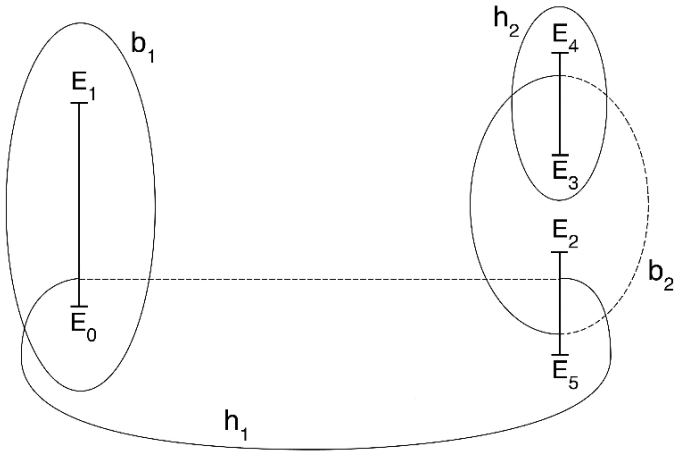


FIG. 2: The three-sheeted Riemann surface with branch points and cycles labelled.

To find the dependence of the $u_i$ on time, it is best to make use of the connection of integrable systems and Riemann surfaces. A central theme in the study of Riemann surfaces are the cycles, which are the non-trivial closed curves on the Riemann surface (those that cannot be smoothly shrunk to a point). They will be denoted by $b_k$ and $h_k$, $k = 1, ..., g$. These cycles are depicted in Fig. 2. Another mainstay of the theory of Riemann surfaces are the Abelian (meromorphic) differentials on the surface. The so-called differentials of the first kind, which are everywhere holomorphic, form a $g$ dimensional vector space for which a basis is:

$$\hat{u}_k = \frac{du}{\sqrt{Q(u)}} u^{k-1} \quad k = 1, \ldots, g. \tag{22}$$

The differentials may be integrated around the non-trivial cycles to assist in the following definitions:

$$\omega_{k,l} = \frac{1}{2} \oint_{b_k} \hat{u}_l, \quad \omega'_{k,l} = \frac{1}{2} \oint_{h_k} \hat{u}_l, \quad \tau = \omega^{-1}\omega'. \tag{23}$$

A familiar construction for genus one Riemann surfaces, which have the topology of a torus, allows us to represent the surface as a rectangle with opposite edges identified. The rectangle is characterized by its aspect ratio, which is an invariant of the Riemann surface as well. In the case of genus one the aspect ratio of the rectangle turns out to be equal to $-i\tau$ from (23). When the genus is higher than one we encounter a matrix, $\tau$, which is a generalization of the number $\tau$ of the genus-1 case. The rectangle with opposite sides identified also has an analogue for higher genera: it is replaced by $2g$ (real) dimensional volume in $\mathbb{C}^g$ given by $\mathbb{C}^g/(\mathbb{Z}^g\omega + \mathbb{Z}^g\omega')$. This $2g$ dimensional volume is an analog of the 2 dimensional rectangle in the genus-1 case, in that there exists an invertible map taking sets of points on the Riemann surface into it. This is given by $\vec{J}(\{u\}) : \Omega \to \mathbb{C}^g/(\mathbb{Z}^g\omega + \mathbb{Z}^g\omega')$, where $\Omega$ is the space of sets of $g$ points $\{u_1, \ldots, u_g\}$ (these are not ordered sets, so permutations are considered equivalent):

$$J_j\left(\{u_i\}_{i=1}^g\right) = \sum_{i=1}^{g} \int_{P_0}^{u_i} d\hat{u}_j. \tag{24}$$

The space $\mathbb{C}^g/(\mathbb{Z}^g\omega + \mathbb{Z}^g\omega')$ is called the Jacobian. The contour of integration from the arbitrary initial point $P_0$ to the point $u$ can wind around any of the cycles any number of times, so as a mapping to $\mathbb{C}^g$ it is only defined up to the addition of an element of the lattice $\mathbb{Z}^g\omega + \mathbb{Z}^g\omega'$. This is however enough to give a well-defined map to $\mathbb{C}^g/(\mathbb{Z}^g\omega + \mathbb{Z}^g\omega')$. One can check that in the case of genus one the mapping takes a point on the Riemann surfaces and maps it onto a rectangle whose aspect ratio is $-i\tau$, the rectangle being represented by $\mathbb{C}/(\mathbb{Z}\omega + \mathbb{Z}\omega')$.

The concept of the Jacobian is particularly important in the solution of the problem because one can show that the zeros of $B(u)$, which are denoted by $u_i$, satisfy the equation:

$$\vec{J}(\{u_i(t)\}_{i=1}^g) = (c_1, c_2, \ldots, c_g + 2it) \tag{25}$$

where we have now written the time dependence of $u_i$ explicitly, while the $c_i$ are defined by:

$$c_i = \sum_{j=1}^{g} \int_{E_{2j}}^{u_j(t=0)} d\hat{u}_i. \tag{26}$$

The roots $E_i$ of $Q(u)$ are listed for our case in (32). Considered as constants of integration for the dynamics of the $g$ roots $u_i(t)$ of $B(u)$, the $c_i$ are $g$ free complex variables determining the initial values $u_i(0)$. However, not all initial values are permissible, i.e. an arbitrary $\vec{c}$ will not correspond to a configuration of the spins; there are $g$ constraints on $\vec{c}$.

The $u_i$ together with the spectral curve $Q(u)$ contain all the information needed to find the configuration of the spins. The problem of finding the $u_i$ is thus the problem of inverting the map $\vec{J}$. This is a solved mathematical problem with a long history, which goes by the name 'the Jacobi Inversion problem' [30]. The solution can be obtained in terms of the Riemann $\theta$-function. We are interested here in the order parameter, whose logarithmic derivative in time is given by $\sum_i u_i$. The explicit solution of the inversion problem for BCS dynamics is presented in [18], and gives the time dependence of the order parameter as:

$$\Delta(t) = \frac{\lambda}{2}\sum_{i=1}^{g+1} s_-^{(i)} = C\exp\left(2\vec{d}\eta(\omega^T)^{-1}\vec{x} - i\beta t\right)\frac{\theta\left((2\omega^T)^{-1}\left(\vec{x}+\vec{d}\right)|\tau\right)}{\theta\left((2\omega^T)^{-1}\left(\vec{x}-\vec{d}\right)|\tau\right)} \tag{27}$$

provided that $\eta$ and $d$ are given by:

$$\eta_{k,l} = -\sum_{j=l+1}^{2g+2-l}\frac{j-l}{4(j+l)!}\frac{d^{j+l}Q(u)}{du^{j+l}}\omega_{k,j}, \qquad d_j = \int_{E_0}^{\infty} d\hat{u}_j. \tag{28}$$

The frequency $\beta$ can be written in terms of the roots $E_i$ of $Q(u)$ as $\beta = 2\sum_i E_i$.

## IV. SOLUTION WHERE IMBALANCE IS PRESENT

In the stationary (but unstable) imbalanced state, $s_z(\xi)$ and $s_-(\xi)$ are given by the expressions (9) and (10). In this case, the eigenvalue $v(u)$ is given by:

$$v^2(u-\Phi) = (u^2+\Delta^2)G(u)^2. \tag{29}$$

Throughout we use $\Delta$ (as opposed to $\Delta(t)$) to denote the value of the gap before the perturbation. $G(u)$, which was defined in (14) because it appeared in the linear stability analysis, has appeared again as a common factor in $\det\mathcal{L}(u) = -A^2(u) - B(u)B^*(u)$. This is not a coincidence – the connection between roots of $v(u)$ and modes present in the solution is explained further in [28]. Since $v(u)^2$ has six roots we are led to consider the three spin problem, which according to the arguments of the previous section represents the dynamics of the order parameter after an initial transient. The polynomial $Q(u)$ for the three-spin problem has the same roots as $v(u)^2$, given thus by:

$$Q(u) = (u^2+\Delta^2)(u-\Phi-i\gamma)^2(u-\Phi+i\gamma)^2. \tag{30}$$

(Here we have shifted $u$ by $\Phi$ – from the equations of motion in [18] we see that this can be compensated by giving the order parameter an additional phase factor.) $Q(u)$ has a very particular structure, related to the fact that it is a stationary solution. It has single roots only at $\pm i\Delta$ and the rest of its roots are double roots. This insures that the Riemann surface given by $Q(u) = y^2(u)$ is of genus 0.

We now wish to perturb the initial conditions. Unless we fine-tune the perturbation to avoid doing so, we will lift the degeneracy associated with the double roots: the polynomial $Q(u)$ will now have 6 single roots (which must still occur in complex conjugate pairs, since $Q$ has real coefficients):

$$Q(u) = (u^2+\Delta^2)\left((u-\Phi-i\gamma)^2+\delta^2\right)\left((u-\Phi+i\gamma)^2+\delta^2\right). \tag{31}$$

For the integrals in (26), (28) we need the definitions (Fig. 2):

$$(E_0,E_1,E_2,E_3,E_4,E_5) = (-i\Delta, i\Delta, \Phi-i\gamma+i\delta, \Phi+i\gamma-i\delta, \Phi+i\gamma+i\delta, \Phi-i\gamma-i\delta). \tag{32}$$

We neglect here (for example) the small shift in the position of the pair of roots around the origin: whereas the splitting of the double roots has a qualitative effect, such shifts have negligible effect on the solution, involving only small shifts in the parameters $\Delta$, $\gamma$ and $\Phi$, and a small change in the overall rate at which the phase of the order parameter rotates. Also, while in general $\delta$ can be complex, in the regime of interest to us the solution is not sensitive to the phase of $\delta$ except through the value of $\vec{c}$. In the following we treat $\delta$ as real unless otherwise stated.

After the perturbation the Riemann surface is of genus two and $\Delta(t)$ is given by (27) with genus two hyperelliptic $\theta$-functions. The expression is quite formidable, yet certain features can be clarified without much analysis. Most notably, the solution is quasi-periodic, with quasi-periods which can be deduced straightforwardly from the general periodicity properties of $\theta$ -functions together with the particular form the matrices $\omega$ and $\tau$ take in this case.

Before continuing to the analysis of the small perturbation case, we note that if the perturbation is large enough it can lead to the appearance of new roots for $Q(u)$, and to higher spin solutions (with higher genera) – this case is too general for us to say much about.

We assume that the parameter $\delta$ is small, as discussed in Section V. We then take the leading order in $\delta$ of the expressions for the matrices $\tau$ and $(2\omega^T)^{-1}(\vec{x}\pm\vec{d})$, which figure in (27). We also expand in $s=\sqrt{(T_c-T)/T_c}$, taking the lowest order terms for each element. Then we have, to leading order (in practice we must make sure lower order terms do not contribute):

$$\omega^{-1}=\begin{pmatrix} -\frac{i\Phi^2}{\pi} & \frac{i\gamma\Delta^2}{\Phi}\log^{-1}\left(\frac{4\gamma}{\delta}\right) \\ \frac{i\Phi}{\pi} & i\gamma\log^{-1}\left(\frac{4\gamma}{\delta}\right)\end{pmatrix};\qquad \eta=\begin{pmatrix} 0 & 0 \\ -\frac{i\pi\Delta^2}{2\Phi} & \frac{i\Phi^2}{2\gamma}\log\left(\frac{\delta}{4\gamma}\right)\end{pmatrix};$$

$$\tau=\begin{pmatrix} \frac{i}{2\pi}\log\left(\frac{4\Phi^4}{\gamma\delta\Delta^2}\right) & -\frac{1}{2}+\frac{i\gamma}{\Phi}\log^{-1}\left(\frac{4\gamma}{\delta}\right) \\ -\frac{1}{2}+\frac{i\gamma}{\Phi}\log^{-1}\left(\frac{4\gamma}{\delta}\right) & \frac{i\pi}{2}\log^{-1}\left(\frac{4\gamma}{\delta}\right)\end{pmatrix};$$

$$(2\omega^T)^{-1}(\vec{x}\pm\vec{d})=\left(-\frac{\Phi t}{\pi}\pm\frac{i}{2\pi}\log\left(\frac{i\Delta}{2\Phi}\right),\left(\gamma t\pm\frac{i\gamma}{2\Phi}\right)\log^{-1}\left(\frac{\delta}{4\gamma}\right)\right)+(2\omega^T)^{-1}\vec{c}. \tag{33}$$

We will return to the vector $\vec{c}$, which requires further analysis. The period matrix $\tau$ is seen to have very large and very small elements on the diagonal, diverging or vanishing with $\log^{\pm1}(\delta)$. Because of this the $\theta$-function is well approximated by trigonometric functions. First we use the modular invariance of $\theta$-functions to trade in our period matrix for one whose elements are all of order $\log(\delta)$. This is done via the identity:

$$\theta\left(\vec{y}\,\middle|\begin{pmatrix} iA & -\kappa \\ -\kappa & ih\end{pmatrix}\right)=\frac{e^{-\frac{\pi}{h}y_2^2}}{\sqrt{h}}\theta\left(\begin{pmatrix} y_1-\frac{i\kappa}{h}y_2 \\ -\frac{i}{h}y_2\end{pmatrix}\middle|\begin{pmatrix} i(A+\kappa^2/h) & i\kappa/h \\ i\kappa/h & i/h\end{pmatrix}\right) \tag{34}$$

which results in a 'transformed' $\tau$-matrix,

$$\tau_{tr}=\frac{i}{\pi}\begin{pmatrix} \log\left(\frac{4\Phi^2}{\delta\Delta}\right) & \log\left(\frac{4\gamma}{\delta}\right) \\ \log\left(\frac{4\gamma}{\delta}\right) & 2\log\left(\frac{4\gamma}{\delta}\right)\end{pmatrix}+\frac{2\gamma}{\pi\Phi}\begin{pmatrix} 1 & 1 \\ 1 & 0\end{pmatrix}. \tag{35}$$

Once the $\theta$-function has been written in terms of a $\tau$ matrix with only large elements, its degeneration into trigonometric functions is easily obtained from the definition of the $\theta$-function in terms of an infinite sum,

$$\theta\left(\vec{u}|\tau\right)\equiv\sum_{\vec{m}\in\mathbb{Z}^g}e^{i\pi(\vec{m}\cdot\tau\vec{m}+2\vec{m}\cdot\vec{u})}. \tag{36}$$

The dominant terms in the sum will come from the $\vec{m}$ close to the the stationary point of the real part of the exponent:

$$\vec{m}_0=-(\mathrm{Im}\tau)^{-1}\mathrm{Im}\vec{u} \tag{37}$$

($\vec{m}_0$ is not necessarily a vector of integers). When $\vec{m}=(m,n)$ deviates much from $\vec{m}_0$ the exponential becomes rapidly smaller because of the largeness of $\tau$, such that the sum is dominated by only a few exponential terms – a (hyper-) trigonometric polynomial. Because $\tau$ is logarithmic in $\delta$ and appears linearly in the exponentials, the sub-dominant terms in the sum will be suppressed by powers of $\delta$.

### A. Recovering the Cooper Instability

By setting $\Phi=\Delta_0/\sqrt{2}$ we completely suppress the gap, returning the system to the normal-metal state. In this limit we should see the absolute value of the order parameter execute the train of $\cosh^{-1}$ solitons found in previous work on the Cooper instability [13]. This behaviour corresponds to a simpler two-spin solution. This degeneration into a system described by fewer spins (or lower-genus $\theta$-functions) is expected whenever we close a branch cut on the Riemann surface for $Q(u)$ (here that joining the roots at $\pm i\Delta$), so long as the initial conditions are such that there

is a $u_i$ pinned at the resulting double root (here $u_1 = 0$) [18]. This is the case when the perturbation is such that $\Delta \ll \delta$, as is appropriate if the perturbation is most significant near the Fermi surface[37]. Note that here $\Delta \neq \Delta(0)$, since the former does not include the effect of the perturbation. With these initial conditions, an appropriate vector of constants $\vec{c}$ is given by

$$\vec{c} = \left( \frac{i}{2\gamma\Phi} \log\left(\frac{4\gamma}{\delta}\right) - \frac{1}{2\Phi^2}\left(\log\left(\frac{4\Phi^4}{\gamma\delta\Delta^2}\right) - 2\right),\ \frac{1}{\Phi} \right). \tag{38}$$

Once we have expressions for the theta functions in terms of the 'large' $\tau$-matrix, we extract the asymptotes in the manner described above. Consider one $\theta$-function, say that in the denominator of (27). To begin with we find the stationary point $\vec{m}_0 = (m_0, n_0)$ via (37), and see that while the value of $n_0$ changes with time, $m_0 \simeq -1$ for small $\Delta$. If we take only this $m$, we have a genus-one $\theta$-function as expected on general grounds. The denominator degenerates similarly, and the ratio has a quasiperiod

$$t_0 = \gamma^{-1} \log\left(\frac{4\gamma}{\delta}\right) \tag{39}$$

corresponding to the quasiperiod of the genus two $\theta$- functions. Numerator and denominator each have a stationary value of $n$, which can differ from that given by (37),

$$n_0(m) = -(\mathrm{Im}(\tau_{22}))^{-1}\mathrm{Im}(u_2 + m\tau_{12}) \tag{40}$$

and the argument of each sum is a Gaussian in $n$ whose width is fixed by the $(2,2)$ component of the transformed $\tau$ matrix.

For generic $t$ we can ignore all but one $n$ for the denominator, giving a single exponential, but when the stationary value of $n$ is close to halfway between two integers two values of $n$ are of comparable importance, yielding an expression for $\theta$ in terms of a trigonometric function. (The numerator behaves similarly half a period later, but at these points the ratio is insignificantly small.) The end result is a train of solitons separated by $t_0$, each of the form (to leading order):

$$\Delta(t) = \frac{2\gamma}{\cosh(2\gamma t)}. \tag{41}$$

The vector $\vec{c}$ and the overall normalization are fixed in the following way. The values of the $u_i$ (including $u_1 = 0$) determine $B(u)$ up to $\Delta(0)$: $B(u) = \frac{2\Delta(0)}{\lambda} u(u - u_2)$. Equation (30) gives $Q(u)$ in the $\Delta \to 0$ limit. Writing $Q(u)$ in terms of $A(u)$ and $B(u)$, we see that $A(u) = \frac{2}{\lambda} u(u - w_1)(u - w_2)$ for some $w_i$ that are either both real, or conjugate to each other. Matching the coefficients of $Q(u)$ written in terms of $A(u)$ and $B(u)$ with (30) gives a family of acceptable solutions for $u_2(0)$ and $\Delta(0)$, corresponding to different stages in the time evolution. Choosing a particular $u_2(0)$ allows us to integrate to get the $\vec{c}$ above, and also fixes $\Delta(0)$ and thus the normalization of our solution. In the above formula we have omitted an overall phase rotation which corresponds to a redefinition of the chemical potential.

The form of the above soliton conforms exactly with previous results [13, 16] for the oscillations of the order parameter following a sudden turn-on of the BCS interaction. Interestingly, it also conforms exactly with the result of the next section, where we assume $\delta \ll \Delta$, in the limit that $\Delta \ll \gamma$. This is despite the difference in the relative sizes of $\Delta$ and $\delta$ in the two cases, which implies very different initial conditions for the $u_i$.

## B. Main Case

We now consider the case $\delta \ll \gamma, \Delta$.

### 1. *Initial Conditions for Small Perturbations*

An important difference between this case and the previous is the value of $\vec{c}$, which (26) gives in terms of the initial positions of our variables $u_i(t = 0)$. These are the zeroes of $B(u)$ at time $t = 0$, which in the unperturbed case coincide with the zeroes of $G(u)$ as one can show using the self-consistency equation. For a small perturbation, the $u_i$ remain close to the branch cuts at $\Phi \pm i\gamma$; let us call them $u_\pm$. It is useful to define $z_\pm$ as the distance of $u_\pm$ from the centers of the branch cuts in units of $\delta$, which may be complex:

$$u_+(t = 0) = \Phi + i\gamma + z_+\delta; \qquad u_-(t = 0) = \Phi - i\gamma + z_-\delta^*. \tag{42}$$

The integrals (26) defining $\vec{c}$ can then be given through $z_\pm$ using the function $I_\pm$ defined as

$$I_\pm = -\int_i^{z_\pm} \frac{\mathrm{d}w}{\sqrt{w^2+1}} = \frac{i\pi}{2} - \operatorname{arcsinh}(z_\pm). \tag{43}$$

The expression for $c_i$ is then given by[38]:

$$c_1 = \frac{I_+\left(1-i\frac{\gamma}{\Phi}\right) - I_-\left(1+i\frac{\gamma}{\Phi}\right)}{2i\gamma\Phi}, \qquad c_2 = \frac{I_+ - I_-}{2i\gamma}; \tag{44}$$

up to corrections suppressed by $\gamma^2/\Phi^2$ and $z\delta/\gamma$.

The form of the solution is sensitive to the values of $c_1$ and $c_2$. But as noted above, these are not independent parameters. The necessary constraints can be found in the following way. We first find expressions for $z_\pm$ in terms of the perturbations to $A(u)$ and $B(u)$. Let $u_I$ be the position of a root of $B(u)$ before the perturbation is added, and $u_F$ its position afterward. Expanding $B = B_0 + \delta B$ about the initial position of the root $u_I$ tells us that, due to the perturbation, $u$ travels a distance given by

$$(u_F - u_I) \simeq -\delta B(u_I)/B_0'(u_I). \tag{45}$$

Similarly we can expand $Q \propto A(u)^2 + B(u)B^*(u)$, taking into account the fact that it has a double root to begin with, and the fact that before the perturbation $A_0$ and $B_0$ are related by $B_0(u) = \frac{\Delta}{\Phi+u}A_0(u)$. This yields both the width of the branch cut (i.e. $2\delta$ or $2\delta^*$) that is opened up by the perturbation, and the location of its centre, in terms of $\delta B(u_I)$ and $\delta A(u_I)$. We omit these formulas. Then the $z_\pm$ are given by

$$z = \frac{(u_F - u_I) - (\text{displacement of centre of branch cut})}{(\text{complex half-length of branch cut})}. \tag{46}$$

Defining $\delta B(u) = \frac{\Delta}{u}\delta A(u) + \delta C(u)$, all the $\delta A$s and $\delta B$s disappear in favor of $\delta C$s, and we can expand without worrying about the relative size of $\delta A$ versus $\delta B$:

$$z_\pm = \frac{\Phi \pm i\gamma}{\Delta}\sqrt{\frac{\delta C(\Phi \pm i\gamma)}{\delta C^*(\Phi \pm i\gamma)}}. \tag{47}$$

Since $\delta C^*(\Phi + i\gamma) = \delta C(\Phi - i\gamma)^*$, this tells us that $|z_+ z_-| = \Phi^2/\Delta^2$ to leading order, and that[39] $\arg z_+/z_-$ is of order $\gamma/\Phi$. Eq. (47) yields the following constraints on $c_i$ or $z_\pm$:

$$p_1 \equiv \mathrm{Re}\ (c_1\Phi^2 - c_2\Phi) = \frac{1}{2}\log 4|z_+ z_-| = \log\frac{2\Phi}{\Delta}, \tag{48}$$

$$p_2 \equiv \mathrm{Re}\ (\gamma c_2) = \frac{1}{2}\arg\frac{z_-}{z_+} \simeq 0. \tag{49}$$

These combinations of $c_1$ and $c_2$ are precisely those necessary for the correct expansion of the $\theta$-functions – for example $p_1$ dictates which integers $(m,n)$ give the leading order contributions to the representation of the theta function as a sum (36).

Having derived these results by expanding $B(u)$, $A(u)$ and $Q(u)$, we must ask when they are valid. Assuming that $\sqrt{\delta C(u_I)/\delta C^*(u_I)}$ is approximately of order one, we find that the roots of $B(u)$ move a distance of order $\delta\Phi/\Delta$. This quantity must be much smaller than $\gamma$, the scale on which our initial polynomials vary. So a necessary condition for the validity of these approximations is

$$\delta \ll \gamma\Delta/\Phi. \tag{50}$$

This excludes of course the Cooper instability case, where one root of $B(u)$ is a distance of order $\Phi$ from the start-points of the integrals in (43). Since $B(u)$ vanishes as $\Delta \to 0$, in this case it is not legitimate to assume that $\delta B \ll B_0$.

### *2. Form of the Solution*

Again we use the 'transformed' $\theta$-functions, and extract the dominant exponentials from the sums defining them, (36). The precise values of $\vec{c}$ depend on the nature of the perturbation, but the information obtained above is enough

to establish the nature of the solution up to (a) an overall shift in time and (b) a shift of the fast oscillations within their envelope. Up to such a shift, each soliton has the form (we neglect $\gamma/\Phi$ corrections):

$$\Delta(t) = \Delta\left(1 + \frac{2\gamma}{\Delta} e^{-2i\Phi t}\mathrm{sech}(2\gamma t)\right) \tag{51}$$

and solitons occur at intervals of $t_0 = \gamma^{-1}\log(4\gamma/\delta)$.

More explicitly, taking into account the expressions for the shifts in terms of $\vec{c}$, the first soliton has the form

$$\Delta(t) = \Delta\left(1 - \frac{2i\gamma}{\Delta}\exp\left(-2i\Phi t - c_2\Phi + c_1\Phi^2 + \log\frac{\Delta}{2\Phi} + i\arg\delta\right)\mathrm{sech}\left(2\gamma(t - t_0/2) - i\gamma c_2\right)\right). \tag{52}$$

From (43,47), Im $c_2$ is of order $\gamma^{-1}\log(\Phi/\Delta)$, so that if $\delta$ is sufficiently small the first soliton takes about half a period to appear.

## V. CONCLUSION

We have found the short-time behavior of a BCS superconductor following a small perturbation to the imbalanced initial conditions given by Eq. 10. These initial conditions show a suppression of the gap [5] with increasing imbalance $\Phi$, and an instability [6] when $\Phi > \Phi_c$ which becomes the celebrated Cooper instability when the gap is fully suppressed. As $\Phi$ is increased beyond $\Phi_c$, the gap oscillations following upon the instability grow in magnitude. They take the form of a train of solitons, each of duration $\sim \gamma^{-1}$ and magnitude $\sim \gamma$, and containing oscillations on the shorter timescale $\Phi^{-1}$ (Eq. 51, and pictured in Fig. 2). These oscillations should be observable if appropriate initial conditions can be prepared in a controlled fashion.

A stronger motivation for the work is that the oscillatory behavior is relevant to evolution on longer time scales ($\sim \tau_\epsilon$) in experimental situations with large imbalance. In particular, the oscillations are relevant to the relaxation of the imbalance, which in the absence of the instability occurs at a rate which vanishes with the gap as $T \to T_c$ [3, 9]. Understanding the short-time dynamics of (5) is a first step; to determine quantitatively what happens on long time scales it is necessary to compute how collisions modulate them. The moduli of the solution, i.e. the variables used to parameterize the kinds of short time behavior, can be taken to be the roots of $Q(u)$. These roots or moduli vary slowly with time on account of collisions. The non-equilibrium state of the system at long times may correspond for example either to an unchanging set of moduli, or to a limit cycle in moduli space. Such an analysis is beyond the scope of the current paper, but will involve (51) and possibly generalizations.

Our explicit expressions for the behavior of the order parameter apply when $\delta \ll \gamma$. In this limit, where the solitons are widely spaced, the expressions simplify greatly; but the generalization to larger $\delta$ may be necessary to treat imbalance relaxation in realistic situations. As an idealized Gendankenexperiment, our limit can be realized by instantaneously injecting electrons at the Fermi level to a system at the instability point – it can be shown using the definition (20) that such a perturbation increases the instability rate $\gamma$ while hardly increasing $\delta$[40]. (A similar analysis shows that in a system with tunable interaction strength, increasing the coupling of a system with $\Phi \lesssim \Phi_c$ does the same.) The subsequent evolution of such systems on time scales $\tau_\epsilon$ may increase $\delta$ further.

The present work neglects spatial inhomogeneities. Whether they change the picture qualitatively in systems larger than the coherence length remains to be investigated. Gap oscillations can also parametrically excite inhomogeneous modes, as shown in [31] for the Cooper instability.

We have already mentioned that thermal processes act by slowly perturbing the dynamics considered. We have not mentioned thermal fluctuations in the initial conditions, which were analysed for the Cooper instability in [16, 32]. These fluctuations disappear when the level spacing, or the effective level spacing in a coherence length, goes to zero, but will cause variations in the parameters of the solution (e.g. our $\delta$) when it is finite. While the evolution is of the same form in each realization, it was shown that variations in the parameters between different realizations are qualitatively important for averages (e.g. of the absolute value of the gap) over realizations. Such averages would be relevant to experiments involving direct observation of gap fluctuations.

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[33] Population imbalance, i.e. having more spin-up than spin-down electrons, may be more easily created. The dynamics for this distinct situation is addressed in [27].
[34] These initial conditions (which will later be perturbed) are thermal averages. Typical initial conditions, as opposed to averaged initial conditions, need not have an $n(\xi)$ that is smooth on the scale of the level spacing. Justification for the use of thermally averaged initial conditions is given in [16]: we can average our spins over a small energy range containing many levels to give a smooth $n(\xi)$ without changing the equations of motion (5, 7).
[35] To obtain (11) one solves the self-consistency equations with (10). The self-consistency condition is dominated by spins in the energy range $|\xi + \Phi| \sim T_C$, where (10) is valid.
[36] In the case where the spectrum is discrete only a finite number of these constants of motion are actually independent.
[37] To see this, consider the initial condition as deriving from a one-spin solution in which $\delta = \Delta = 0$ (here all the $s_-^{(i)} = 0$, so that $B = 0$) by a perturbation which opens the branch cuts of size $\delta$ and $\Delta$. After the perturbation, the (three-spin) $B(u)$ is $\sim \delta(u - w_0)(u - w_1)$ for some $w_{0,1}$. Deducing the size of the branch cuts in terms of $w_0$ and $w_1$ from the expression for $Q(u)$ in terms of $A(u)$ and $B(u)$, and demanding that $\Delta \ll \delta$, we see that one of the $w$ is close to the $\Delta$-sized cut and one is close to the $\delta$-sized cuts; as $\Delta \to 0$, the former root is pinned to the origin.
[38] The minus sign in front of the integral in (43) is due to the fact that the $u_i$ have to be chosen on the lower Riemann sheet in this case. This can be ascertained by examination of the final result once derived.
[39] For this last we must resolve a sign ambiguity coming from the square roots, which is most easily done by examining the special case where $\delta$ is real and the centers of the branch cuts do not move.
[40] What is important is the structure of the Riemann surface given by the spectral polynomial (20), rather than the details of the distribution of the spins.